A New Semi-Analytical Solution of the Telegraph Equation with Integral Condition

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In the current work, the telegraph equation in its general form and with an integral condition is investigated. Also the well-known homotopy analysis method (HAM) is applied and an interesting iterative algorithm is proposed for solving the problem in general form. Some numerical examples are given and compared with the exact solution to show the effectiveness of the proposed method.

Key words: Telegraph Equation; Linear and Nonlinear Forcing; Nonlocal Boundary Condition; Series Solution; Integral Condition; Homotopy Analysis Method.

1. Introduction

Boundary value problems with integral conditions constitute a very interesting and important class of problems which are widely used for mathematical modelling of various processes of physics, ecology, chemistry, biology, and industry [1 – 6], when it is impossible to determine the boundary or initial values of the unknown function. The presence of an integral term in a boundary condition causes that the theoretical study of nonlocal problems is connected with great difficulties and also the application of many standard numerical techniques such as finite difference, finite elements, spectral methods, and so on, for solving these types of problems can be greatly complicated. Therefore, to apply them widely to problems of practical interests, in general, it is important to convert nonlocal boundary value problems into more desirable forms. In recent years, several numerical techniques have been presented to solve various types of nonlocal boundary value problems [7 – 14].

In this paper, we consider the nonlocal boundary value problem for the telegraph equation

$$\begin{aligned}&\frac{\partial^2 v}{\partial t^2}+\alpha\frac{\partial v}{\partial t}=\beta^2\frac{\partial^2 v}{\partial x^2}+F(v,x,t),\\ &(x,t)\in\Omega=(0,\ell)\times(0,T],\end{aligned}\tag{1}$$

with the initial conditions

$$\begin{aligned}v(x,0)&=r(x), \quad 0\le x\le\ell,\\ v_t(x,0)&=s(x), \quad 0\le x\le\ell,\end{aligned}\tag{2}$$

the Neumann condition

$$v_x(0,t)=p(t),\ \ 0<t\le T,\tag{3}$$

and the integral (nonlocal) condition

$$\int_0^\ell v(x,t)dx=q(t),\ \ 0<t\le T,\tag{4}$$

where $F(v,x,t)$, r, s, p, and q are given functions, and $\alpha>0$, $\ell>0$, $T>0$, and $\beta\in\mathbb{R}$. Note that the force $F(v,x,t)$ can be a linear or nonlinear function. In [15], the authors investigated this problem and discussed the existence and uniqueness of the solution of this important problem by using the Rothe method. In [16], the author proposed an iterative method for solving this problem. Recently, Salkuyeh and Roohani in [17] applied the variational iteration method for solving an special case of this problem (the problem with linear forcing). Throughout these papers, we assume that F is sufficiently smooth to produce a smooth classical solution v. Here we mention that the functions r and s

satisfy the following compatibility conditions:

$$r'(0) = p(0), \qquad \int_0^{\ell} r(x)\,\mathrm{d}x = q(0),$$
$$s'(0) = p'(0), \qquad \int_0^{\ell} s(x)\,\mathrm{d}x = q'(0).$$

As we know, the homotopy analysis method (HAM) [18–24] is a powerful device for solving differential equations. This method has been applied successfully to solve many problems of various fields of science and engineering. Recently, Mohyud-Din and Yıldırım in [25] applied HAM for solving two-dimensional diffusion with an integral condition. In the next section, we will present an iterative algorithm based on HAM to solve the problem given in (1)–(4).

2. Homotopy Analysis Method for the Telegraph Equation

For the sake of simplicity and also for finding a systematic algorithm based on the homotopy analysis method, we first transform (1) with the initial conditions (2) and inhomogeneous conditions (3) and (4) to an equivalent one with homogenous conditions. To do so, we use the transformation $u(x,t) = v(x,t) - z(x,t)$ [14, 15], where

$$z(x,t) = p(t)\left(x - \frac{\ell}{2}\right) + \frac{q(t)}{\ell}.$$

In this case, by a simple manipulation the problem is transformed to

$$\frac{\partial^2 u}{\partial t^2} + \alpha\frac{\partial u}{\partial t} = \beta^2\frac{\partial^2 u}{\partial x^2} + \bar{F}(u(x,t),x,t), \quad (x,t) \in \Omega = (0,\ell)\times(0,T], \tag{5}$$

$$u(x,0) = \bar{r}(x), \quad 0 \le x \le \ell, \qquad u_t(x,0) = \bar{s}(x), \quad 0 \le x \le \ell, \tag{6}$$

$$u_x(0,t) = 0, \qquad 0 < t \le T, \tag{7}$$

$$\int_0^{\ell} u(x,t)\,\mathrm{d}x = 0, \ \ 0 < t \le T, \tag{8}$$

where

$$\bar{F}(u(x,t),x,t) = F(u(x,t)+z(x,t),x,t) - \frac{\partial^2 z}{\partial t^2} - \alpha\frac{\partial z}{\partial t},$$
$$\bar{r}(x) = r(x) - z(x,0),$$
$$\bar{s}(x) = s(x) - \frac{\partial z}{\partial t}(x,0).$$

As we observe, the Neumann and the integral conditions are now homogeneous. Hence, instead of looking for v, we simply look for u. We focus our attention on the problem given in (5)–(8). Following the standard procedure of HAM, according to (5) and based on the method of linear partition matching [23], we choose the linear operator

$$\mathcal{L}[\phi(x,t;q)] = \frac{\partial^2\phi(x,t;q)}{\partial t^2} + \alpha\frac{\partial\phi(x,t;q)}{\partial t}$$

and define subsequently the nonlinear operator $\mathcal{N}$ as

$$\mathcal{N}[\phi(x,t;q)] = \frac{\partial^2\phi(x,t;q)}{\partial t^2} + \alpha\frac{\partial\phi(x,t;q)}{\partial t} - \beta^2\frac{\partial^2\phi(x,t;q)}{\partial x^2} - \bar{F}(\phi(x,t;q),x,t). \tag{9}$$

The selection of the initial approximation is one of the most important choices we can make when employing HAM. The initial approximation should satisfy the initial and the boundary conditions of the problem. It is clear that for this problem we can choose the initial approximation in the form $u_0(x,t) = \bar{r}(x) + t\bar{s}(x)$, which satisfies all conditions (6)–(8). Thus based on HAM [18, 19], the general zero-order deformation equation is

$$(1-q)\mathcal{L}[\phi(x,t;q) - u_0(x,t)] = q\hbar\mathcal{H}(x,t)\mathcal{N}[\phi(x,t;q)],$$

with the corresponding initial conditions

$$\phi(x,0;q) = \bar{r}(x),$$
$$\frac{\partial}{\partial t}\phi(x,0;q) = \bar{s}(x),$$

where $q \in [0,1]$ is an embedding parameter, $\hbar \neq 0$ is a nonzero auxiliary parameter, $\mathcal{H} \neq 0$ is an auxiliary function (in this work, we choose $\mathcal{H}(x,t) = 1$ for simplicity), and $\phi(x,t;q)$ is an unknown function. Interestingly, there is a marked great freedom in the choice of auxiliary parameters in HAM. Obviously, when $q = 0$ and $q = 1$,

$$\phi(x,t;0) = u_0(x,t), \ \ \phi(x,t;1) = u(x,t),$$

respectively hold. Thus as q increases from 0 to 1, the solution $\phi(x,t;q)$ varies from the initial guess $u_0(x,t)$ to the exact solution $u(x,t)$. We now expand the function $\phi(x,t;q)$ in a Taylor series to the embedding parameter q in the form

$$\phi(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)q^m,$$

where

$$u_m(x,t) = \frac{1}{m!}\frac{\partial^m \phi(x,t;q)}{\partial q^m}|_{q=0}.$$

If the auxiliary linear operator $\mathcal{L}$, the initial guess u_0, and the auxiliary parameter $\hbar$ are all properly chosen, then the above series converges, when $q = 1$, to end up with

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t).$$

The associated high-order deformation equation is

$$\mathcal{L}[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar(R_m(\vec{u}_{m-1})) \quad (10)$$

with the initial conditions

$$u_m(x,0) = 0,$$
$$u_{t,m}(x,0) = 0,$$

in which

$$R_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathcal{N}[\phi(x,t;q)]}{\partial q^{m-1}}|_{q=0} \quad (11)$$

and

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases}$$

From (11) and (9), we have

$$\begin{aligned} &R_m(\vec{u}_{m-1}) = u_{tt,m-1}(x,t) \\ &+ \alpha u_{t,m-1}(x,t) - \beta^2 u_{xx,m-1}(x,t) \\ &- \frac{1}{(m-1)!}\frac{\partial^{m-1}\bar{F}(\phi(x,t;q),x,t)}{\partial q^{m-1}}|_{q=0}, \end{aligned} \quad (12)$$

where the indexes x and t denote differentiation with respect to x and t, respectively. So, we can present the following systematic iterative algorithm for solving the problem (1)–(4):

$$\begin{aligned} &u_0(x,t) = \bar{r}(x) + t\bar{s}(x), \\ &u_m(x,t) = \chi_m u_{m-1}(x,t) + \mathcal{L}^{-1}[\hbar(R_m(\vec{u}_{m-1}))], \quad m \ge 1, \\ &u_m(x,0) = 0, \quad u_{t,m}(x,0) = 0. \end{aligned} \quad (13)$$

In particular, for the well-known telegraph equation with linear forcing,

$$\frac{\partial^2 v}{\partial t^2} + \alpha\frac{\partial v}{\partial t} = \beta^2\frac{\partial^2 v}{\partial x^2} + \gamma v + F(x,t),$$
$$(x,t) \in \Omega = (0,\ell) \times (0,T],$$

with conditions (2)–(4), we can obtain the following systematic iterative algorithm based on HAM:

$$\begin{aligned} &u_0(x,t) = \bar{r}(x) + t\bar{s}(x), \\ &u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar\mathcal{L}^{-1}[u_{tt,m-1}(x,t) \\ &+ \alpha u_{t,m-1}(x,t) - \beta^2 u_{xx,m-1}(x,t) - \gamma u_{m-1}(x,t) \\ &- (1-\chi_m)\bar{F}(x,t))], \quad m \ge 1, \\ &u_m(x,0) = 0, \quad u_{t,m}(x,0) = 0, \end{aligned} \quad (14)$$

where $\bar{F}(x,t) = F(x,t) - \frac{\partial^2 z}{\partial t^2} - \alpha\frac{\partial z}{\partial t} + \gamma z$. Now we can obtain the Mth-order approximate solution $U_M(x,t) = \sum_{m=0}^{M} u_m(x,t)$ by computing the u_i's from solving the ordinary differential equation presented in (13).

3. Numerical Examples

In this section, we analyze some examples of the problem (1)–(4) and choose proper $\hbar$ with the help of $\hbar$-curves.

Example 1. For the first example, we consider the following nonlinear telegraph equation [16]:

$$\frac{\partial^2}{\partial t^2}v(x,t) + \frac{\partial}{\partial t}v(x,t) = \frac{\partial^2}{\partial x^2}v(x,t) + v^2(x,t) + f(x,t),$$
$$(x,t) \in \Omega = (0,1.0) \times (0,1.0]$$

with

$$\begin{aligned} &f(x,t) = \mathrm{e}^{(-2t)}[-9t^2(1-3x^2)^2 \\ &+ 6\,\mathrm{e}^t(-1+3x^2+6t+3t^2(1-3x^2)^2) \\ &- 3\,\mathrm{e}^{(2t)}(2-6x^2+12t+3t^2(1-3x^2)^2)]/4, \\ &0 \le x \le 1.0, \quad 0 < t \le 1.0, \\ &r(x) = 0, \quad s(x) = 0, \quad 0 \le x \le 1.0, \\ &p(t) = 0, \quad q(t) = 0, \quad 0 < t \le 1.0. \end{aligned}$$

The exact solution of this problem is $v(x,t) = \frac{3}{2}(t\,\mathrm{e}^{-t} - t)(1-3x^2)$. For this problem, we consider

$$\begin{aligned} &\mathcal{L}[v_m(x,t)] = \frac{\partial^2}{\partial t^2}v_m(x,t) + \frac{\partial}{\partial t}v_m(x,t), \\ &R_m(\vec{v}_{m-1}) = v_{tt,m-1}(x,t) + v_{t,m-1}(x,t) - v_{xx,m-1}(x,t) \\ &- \sum_{j=0}^{m-1} v_j(x,t)v_{m-j-1}(x,t) - (1-\chi_m)f(x,t), \end{aligned}$$

and we use $v_0(x,t) = 0$ as the initial guess. Thus from (13), we can obtain an analytical approximate solution for the problem. To assess the impact of $\hbar$ on the convergence of the obtained approximate solution, we

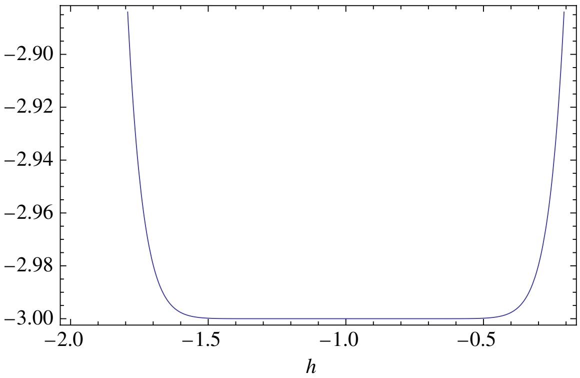

Fig. 1 (colour online). $\hbar$-curve for 14th-order approximate for Example 1; $U_{14,2t}(0,0)$.

first plot the so-called $\hbar$-curve of $U_{14,2t}(0,0)$ in Figure 1. From this figure it is evident that the valid domain of $\hbar$ is $\hbar \in (-1.5,-0.5)$ for the convergence of the series solution. To show the convergence behaviour of the series solutions, the values of $\|v-V_m\|_\infty$ where $V_m = \sum_{i=0}^{m} v_i$ for some valid convergence control parameters $\hbar$ and different values of m are given in Table 1; the last row in this table shows the obtained result by the author of [16]. Clearly, we can observe that the approximate solutions obtained when $\hbar=-1$ are more accurate than the approximate solutions obtained form another choice of $\hbar$. As the numerical results in this table show, the proposed method is very effective for solving this type of difficult problems, and it is evident that the efficiency of HAM can be dramatically enhanced by computing further terms of the truncated series. For more investigation, the absolute error

$$E(x,t) = \{|v(x,t)-V_{14}(x,t)|,\\ (x,t)\in\Omega=(0,1.0)\times(0,1.0]\}$$

for $\hbar=-1$ is plotted in Figure 2. As we observe, there is a very good agreement between the approximate solution obtained by HAM and the exact solution.

Example 2. For the second example, we consider the following nonlinear telegraph equation:

$$\frac{\partial^2}{\partial t^2}v(x,t)+2\frac{\partial}{\partial t}v(x,t)=\frac{\partial^2}{\partial x^2}v(x,t)+F(v(x,t),x,t),\\ (x,t)\in\Omega=(0,2\pi)\times(0,1.0],$$

with

$$F(v,x,t)=v^2-2v\,\mathrm{e}^{-t}(x-\pi)-\mathrm{e}^{-2t}(\sin^2(x)\\ -2\sin(x)(x-\pi)),\\ r(x)=\sin x,\ \ s(x)=-\sin x,\ \ 0\le x\le 2\pi,\\ p(t)=\mathrm{e}^{-t},\ \ q(t)=0,\ \ 0<t\le 1.0.$$

The exact solution of this problem is $v(x,t)=\mathrm{e}^{-t}\sin x$. For this problem, we can obtain

$$z(x,t)=\mathrm{e}^{-t}(x-\pi),\\ \bar{F}(u,x,t)=u^2+f(x,t),\\ f(x,t)=-\mathrm{e}^{-t}(-x+\pi+\mathrm{e}^{-t}\sin(x)^2-2\mathrm{e}^{-t}\sin(x)x\\ +2\mathrm{e}^{-t}\sin(x)\pi+\mathrm{e}^{-t}x^2-2\mathrm{e}^{-t}x\pi+\mathrm{e}^{-t}\pi^2),\\ \bar{r}(x)=\sin x-x+\pi,\\ \bar{s}(x)=-\sin x+x-\pi.$$

For this case, we consider $u_0(x,t)=(1-t)(\sin(x)-x+\pi)$ as the initial guess. Now, similar to the previous example, we can employ HAM and obtain an analytical solution for this example from (13) by using

$$u_0(x,t)=(1-t)(\sin(x)-x+\pi),\\ \mathcal{L}[u_m(x,t)]=\frac{\partial^2}{\partial t^2}u_m(x,t)+2\frac{\partial}{\partial t}u_m(x,t),\\ R_m(\vec{u}_{m-1})=u_{tt,m-1}(x,t)+2u_{t,m-1}(x,t)-u_{xx,m-1}(x,t)\\ -\sum_{j=0}^{m-1}u_j(x,t)u_{m-j-1}(x,t)-(1-\chi_m)f(x,t).$$

To assess the impact of $\hbar$ on the convergence of the obtained approximate solution, we first plot the $\hbar$-curve of $U_{5,2t}(0,0)$ in Figure 3. From this figure it is evident that the valid domain of $\hbar$ is $\hbar\in(-1.2,-0.8)$ for the convergence of the series solution. For more investigation, the absolute error function

$$E(x,t)=\{|v(x,t)-V_5(x,t)|,\\ (x,t)\in\Omega=(0,2\pi)\times(0,1.0]\}$$

m	6	8	10	12	14
$\hbar=-0.9$	8.2819e-3	9.2871e-4	1.5826e-4	1.7111e-5	1.8869e-6
$\hbar=-1.0$	1.7878e-3	1.9267e-4	1.2631e-5	8.9721e-7	6.5211e-8
$\hbar=-1.1$	2.9641e-3	2.4364e-4	4.1228e-5	6.6278e-6	4.2167e-7
$\|v-v_{225}\|_\infty$ presented in [16]$\simeq$			8e-4		

Table 1. Numerical results for Example 1.

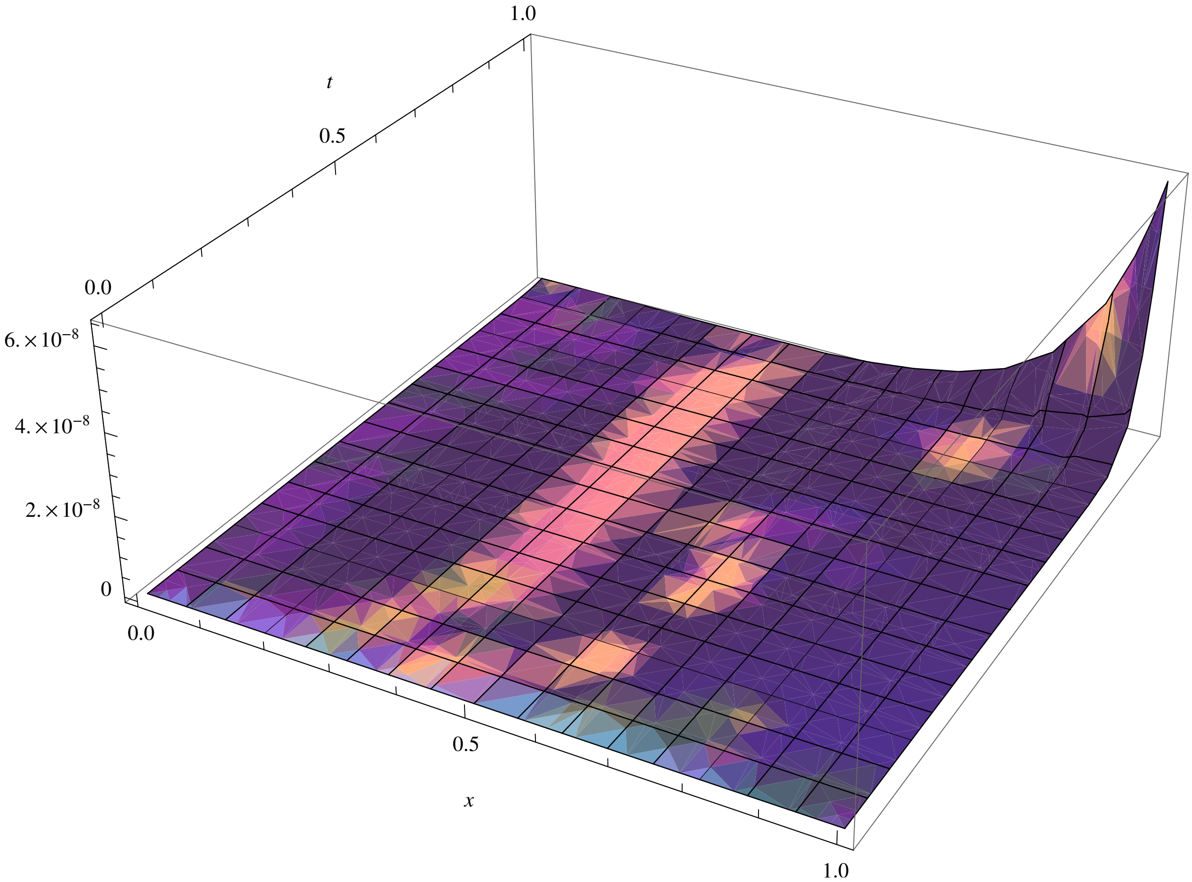

Fig. 2 (colour online). Depiction of the absolute error for Example 1 with $M = 14$ and $\hbar = -1$.

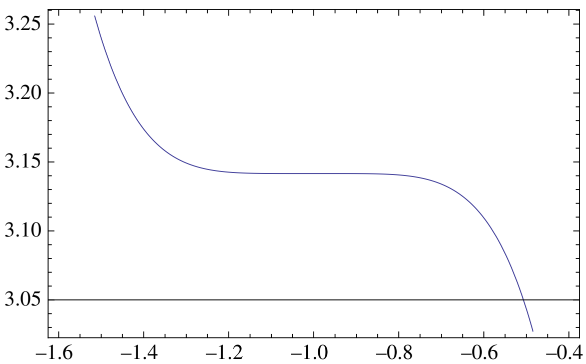

Fig. 3 (colour online). $\hbar$-curve for 5th-order approximate $(u_{5,2t}(0,0))$ for Example 2.

for $\hbar = -1$ is plotted in Figure 4, where $V_5 = U_5 + z$. As we observe, there is a very good agreement between the approximate solution obtained by HAM and the exact solution for this nonlinear equation.

Example 3. For the third example, we consider the following linear telegraph equation:

$$\frac{\partial^2}{\partial t^2}v(x,t) + 12\frac{\partial}{\partial t}v(x,t) = \frac{\partial^2}{\partial x^2}v(x,t) - 4v(x,t) + F(x,t),\ \ (x,t) \in \Omega = (0,\pi) \times (0,1.0],$$

with

$$\begin{aligned} &F(x,t) = 4\sin x(\cos t - 3\sin t),\\ &\qquad 0 \le x \le \pi,\ \ 0 < t \le 1.0,\\ &r(x) = \sin x,\ \ s(x) = 0,\ \ 0 \le x \le \pi,\\ &p(t) = \cos t,\ \ q(t) = 2\cos t,\ \ 0 < t \le 1.0. \end{aligned}$$

The exact solution of this problem is $v(x,t) = \cos t \sin x$ [17, 26, 27]. For this problem, we obtain

$$z(x,t) = \frac{2\pi x - \pi^2 + 4}{2\pi}\cos(t),$$

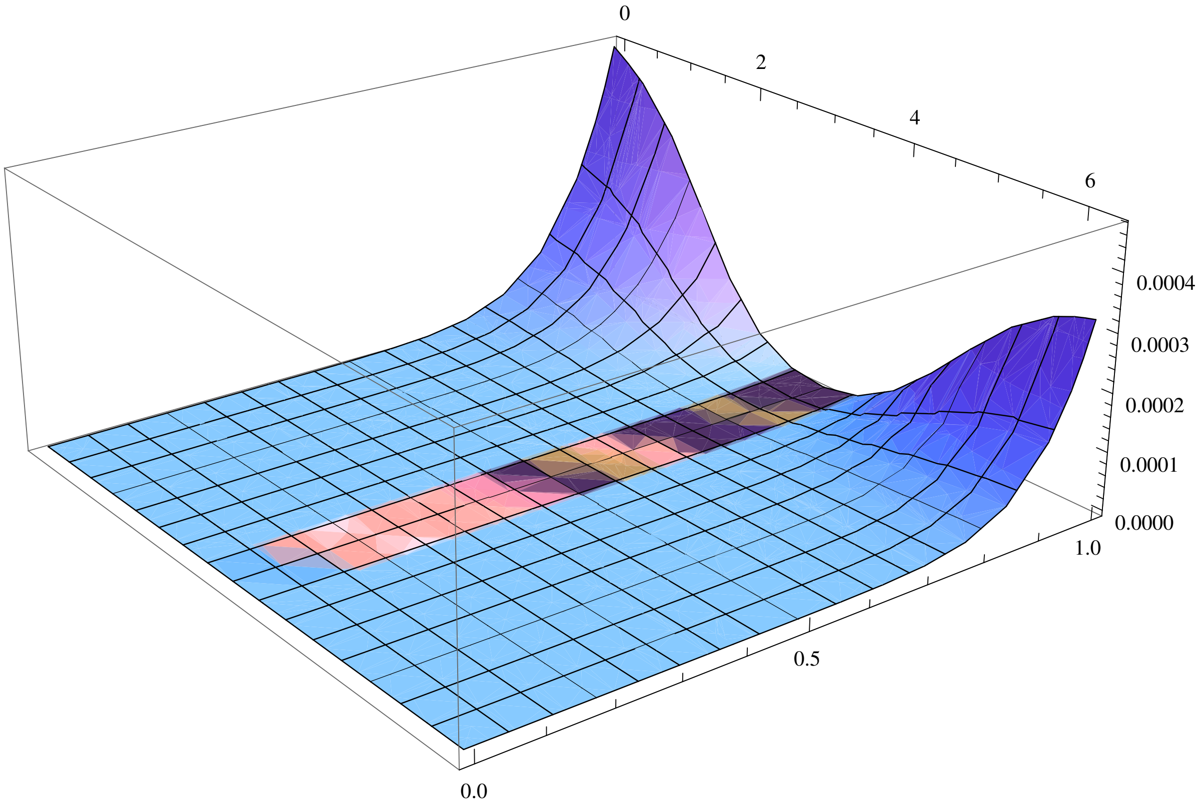

Fig. 4 (colour online). Depiction of the absolute error for Example 2 with $M = 5$ and $\hbar = -1$.

$$\bar{F}(x,t) = 4\sin x(\cos t - 3\sin t) - \frac{3(2\pi x - \pi^2 + 4)}{2\pi}\cos t + \frac{6(2\pi x - \pi^2 + 4)}{\pi}\sin t,$$

$$\bar{r}(x) = \sin x - \frac{2\pi x - \pi^2 + 4}{2\pi},$$

$$\bar{s}(x) = 0.$$

Now, based on the HAM, we can obtain a semi-analytical solution for this problem from (14) by using

$$u_0(x,t) = \sin(x) - \frac{2\pi x - \pi^2 + 4}{2\pi},$$

$$\mathcal{L}[u_m(x,t)] = \frac{\partial^2}{\partial t^2}u_m(x,t) + 12\frac{\partial}{\partial t}u_m(x,t),$$

$$R_m(\vec{u}_{m-1}) = u_{tt,m-1}(x,t) + 12u_{t,m-1}(x,t) - u_{xx,m-1}(x,t) + 4u_{m-1}(x,t) - (1-\chi_m)\bar{F}(x,t).$$

Similar to the previous cases, to assess the impact of $\hbar$ on the convergence of the obtained approximate solution, we first plot the so-called $\hbar$-curve of $U_{10,x,4t}(0,0)$, $U_{10,x,5t}(0,0)$, and $U_{10,x,6t}(0,0)$ in Figure 5. From this figure it is evident that the valid domain of $\hbar$ is $\hbar \in (-1.1, -0.9)$ for the convergence of the series solution. To show the convergence behaviour of the series solutions, the values of $\|v - V_m\|_\infty$ for some valid

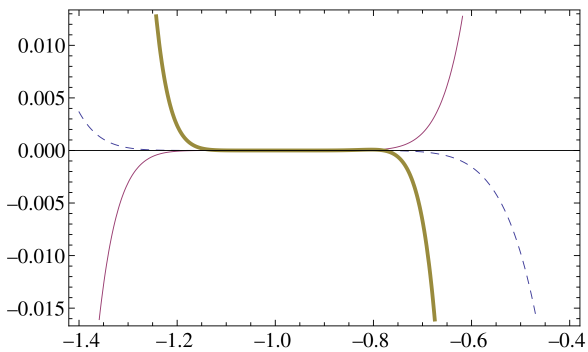

Fig. 5 (colour online). $\hbar$-curve for 10th-order approximate for Example 3; dashed line: $U_{10,x,4t}(0,0)$; bold line: $U_{10,x,5t}(0,0)$; thick line: $U_{10,x,6t}(0,0)$.

m	2	4	6	8	10
$\hbar = -0.9$	1.4371e-3	3.0518e-5	3.6379e-7	2.0586e-9	2.8421e-11
$\hbar = -1.0$	3.0285e-3	6.1178e-6	3.3358e-9	8.0075e-13	8.9599e-17
$\hbar = -1.1$	1.7266e-2	1.0145e-3	2.3786e-5	5.0199e-7	9.8166e-9
Maximum of error in [17]			4.204e-10		

Table 2. Numerical results for Example 3.

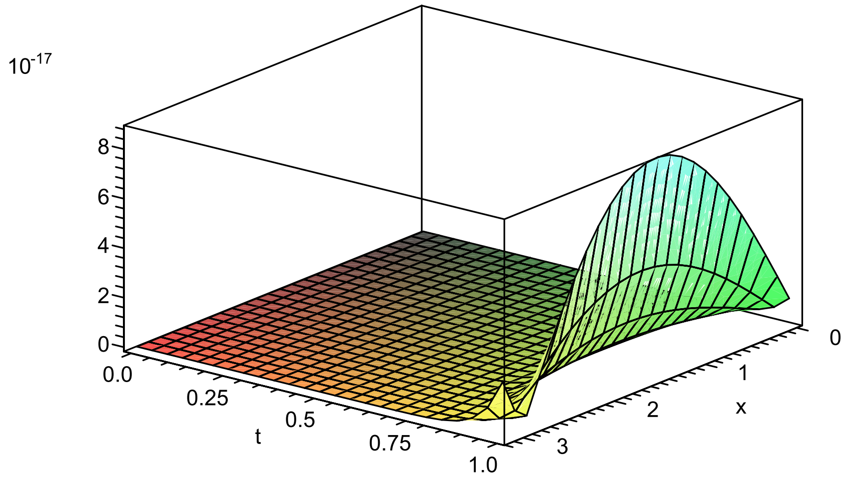

Fig. 6 (colour online). Depiction of the absolute error for Example 3 with $M = 10$ and $\hbar = -1$.

convergence control parameters $\hbar$ and different values of m are given in Table 2. For more investigation, the absolute error

$$E(x,t) = \{|v(x,t) - V_{10}(x,t)|,\\ (x,t) \in \Omega = (0,\pi) \times (0,1.0]\}$$

for $\hbar = -1$ is plotted in Figure 6. From the presented results through the Table 2 and Figure 6 it is evident that there is a very good agreement between the approximate solution obtained by HAM and the exact solution.

Example 4. Consider the problem

$$\frac{\partial^2}{\partial t^2}v(x,t) + 2\frac{\partial}{\partial t}v(x,t) = \frac{\partial^2}{\partial x^2}v(x,t) - v(x,t)\\ + F(x,t), \quad (x,t) \in \Omega = (0,4) \times (0,3],$$

with

$$F(x,t) = 2(t^2 - x - x^2)\,\mathrm{e}^{-t}, \quad 0 \le x \le 4, \quad 0 < t \le 3,$$
$$r(x) = 0, \quad s(x) = 0, \quad 0 \le x \le 4,$$
$$p(t) = -t^2\,\mathrm{e}^{-t}, \quad q(t) = -\frac{88}{3}t^2\,\mathrm{e}^{-t}, \quad 0 < t \le 3.$$

The exact solution of this problem is $v(x,t) = -(x^2 + x)t^2\,\mathrm{e}^{-t}$ [17]. Here, we have

$$z(x,t) = -\frac{1}{3}t^2(3x + 16)\,\mathrm{e}^{-t},$$
$$\bar{F}(x,t) = \frac{2}{3}(-3x^2 + 3t^2 + 16)\,\mathrm{e}^{-t},$$
$$\bar{r}(x) = 0, \quad \bar{s}(x) = 0.$$

For this example, we use $u_0(x,t) = 0$ as the initial guess. Similar to the previous example, we can obtain a semi-analytical solution for this problem from (14). In Table 3 the values of $\|v - V_m\|_\infty$ for some values of m using $\hbar = -1$ are given and in Figure 7 the error function for this problem obtained from 15th-order HAM is plotted, showing the convergence behaviour of the series solutions.

Example 5. For the last example, we consider

$$\frac{\partial^2}{\partial t^2}v(x,t) + 2\frac{\partial}{\partial t}v(x,t) = \frac{\partial^2}{\partial x^2}v(x,t) - v(x,t),\\ (x,t) \in \Omega = (0,4) \times (0,4],$$

with

$$r(x) = \mathrm{e}^x, \quad s(x) = -2\,\mathrm{e}^x, \quad 0 \le x \le 4,$$
$$p(t) = \mathrm{e}^{-2t}, \quad q(t) = \mathrm{e}^{-2t}(\mathrm{e}^4 - 1), \quad 0 < t \le 4.$$

m	5	7	10	13	15
$\|v-V_m\|_\infty$	2.1633e-3	4.8581e-6	8.2827e-11	2.6767e-16	2.7934e-20

Table 3. Numerical results for Example 4 with $\hbar=-1$.

m	5	7	10	12	15
$\|v-V_m\|_\infty$	9.1856e-2	4.4907e-4	2.9100e-8	1.9347e-11	1.1266e-16

Table 4. Numerical results for Example 5 with $\hbar=-1$.

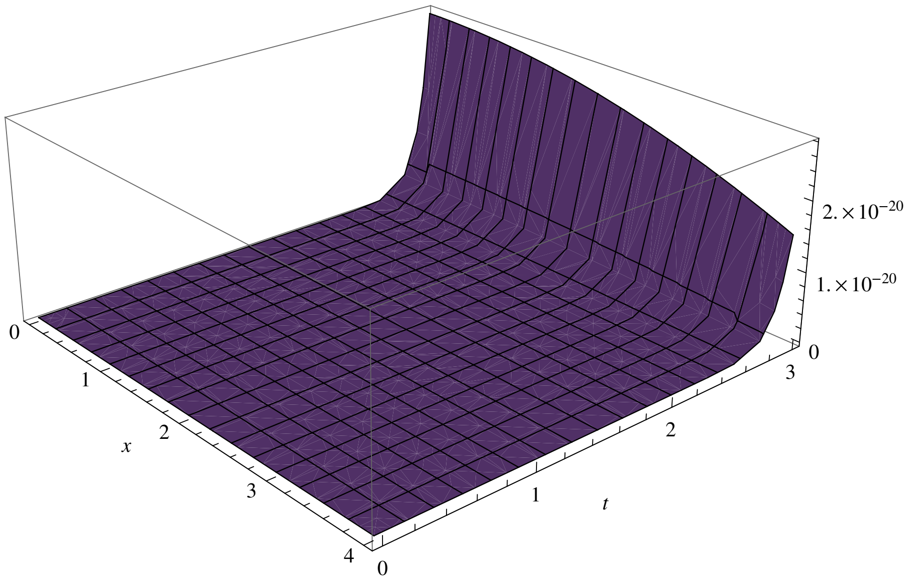

Fig. 7 (colour online). Depiction of the absolute error for Example 4 with $M=15$ and $\hbar=-1$.

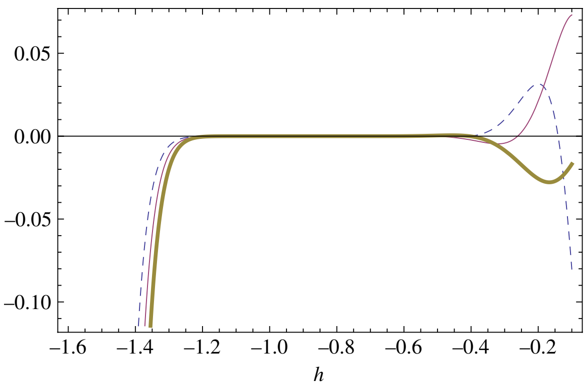

Fig. 8 (colour online). $\hbar$-curve for 15th-order approximate for Example 5; dashed line: $U_{15,2t,x}(0,4)$; bold line: $U_{15,3t,x}(0,4)$; thick line: $U_{10,4t,x}(0,5)$.

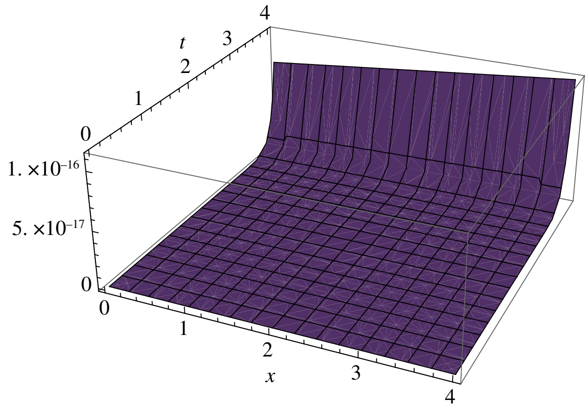

Fig. 9 (colour online). Depiction of the approximate solution for Example 5 with $M=15$ and $\hbar=-1$.

The exact solution of this problem is $v(x,t)=\mathrm{e}^{x-2t}$ [24]. Here, we have

$$z(x,t)=\frac{1}{4}\,\mathrm{e}^{-2t}(\mathrm{e}^4-9+4x),$$
$$\bar{F}(x,t)=-\frac{1}{4}\,\mathrm{e}^{-2t}(\mathrm{e}^4-9+4x),$$
$$\bar{r}(x)=\frac{1}{4}(9-\mathrm{e}^4)+\mathrm{e}^x-x,$$
$$\bar{s}(x)=\frac{1}{2}(\mathrm{e}^4-9+4x-4\,\mathrm{e}^x).$$

Based on the proposed approach, we consider $u_0(x,t)=\frac{1}{4}(2t-1)(\mathrm{e}^4-9+4x-4\,\mathrm{e}^x)$ as the initial guess, employ HAM for this example, and obtain an analytical solution from (14). In Figure 8 the $\hbar$-curves of $U_{15,2t,x}(0,4)$, $U_{15,3t,x}(0,4)$, and $U_{15,4t,x}(0,4)$ are plotted, showing a guideline to assess the impact of $\hbar$ on the convergence of the obtained approximate solution. Based on this figure, we find that the valid domain of $\hbar$ is $\hbar \in (-1.2, -0.6)$. To show the convergence behaviour of the series solutions, the values of $\|v - V_m\|_\infty$ for some values of m with $\hbar = -1$ are given in Table 4 and also the error function is plotted in Figure 9.

4. Conclusions

In this paper, the homotopy analysis method (HAM) has been demonstrated to be applicable in the solution of the telegraph equation in both linear and nonlinear cases with integral condition, and a systematic algorithm for solving this problem has been presented. Equipped with a flexibility in choosing $\hbar$, the HAM exhibits a unique feature for controlling the convergence of the approximation series to the solution of this problem. The results obtained by using HAM are very highly accurate when compared with those results which already exist in the literature. The obtained results presented in this paper show that the proposed method can solve the problem effectively and achieve a good approximate solution with the exact solution.

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